2-Exponential PDF and Analytic Uncertainty Approximations for Rossi-alpha Histograms

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Introduction

• Special nuclear material (SNM) undergoes fission; neutrons from fission can cause another fission…
  → neutron-multiplying system (characterized by reactivity).

• The reactivity of a subcritical system is of interest in:
  – **Nuclear Nonproliferation**: fuel pin diversion? Is a source neutron-multiplying?
  – **Criticality Safety**: will the reactor regain subcriticality during normal and credible upset conditions? In-situ measurements.
  – **Accelerator-Driven Systems**
  – **Emergency Response**: determine if a sample is multiplying or if neutrons are from another source e.g., (alpha,n).

• **Challenge**: we cannot directly estimate a system’s subcritical reactivity.
Background

• In neutron-multiplying (fission-chain) systems, neutron emissions/detections are not uniformly distributed in time.
  – This is due to time-correlation between prompt neutrons originating from the same fission.

• The non-uniformity can be observed by producing a histogram of the times between neutron detections.
  – This is the Rossi-alpha histogram.

• Traditionally, it is assumed the trend is described by:

\[ p(t) = A + Be^{\alpha t} \]

\( \alpha \) is the prompt-neutron decay constant.

• The subcritical reactivity can be inferred from the prompt neutron decay constant, alpha.

• We can estimate alpha by fitting the Rossi-alpha histogram with \( p(t) \).
Prior Work

• It has been shown for detectors with polyethylene moderation that a 2-exp fit is more adequate than an 1-exp fit due to *slowing-down time*.
  – For this presentation, we assume 2-exp is more adequate and focus on the physical meaning/correspondence of the 2-exp.

• A two-region point-kinetics model for the number of neutrons in a fissile core and reflector has been developed.

• Currently, uncertainty in the estimated Rossi-alpha parameter is calculated by taking many measurements/splitting one long measurement and obtaining a sample standard deviation.

\[
\begin{align*}
\frac{dN_c}{dt} &= \frac{k_c - 1}{\ell_c} N_c + f_{rc} \frac{N_r}{\ell_r} \\
\frac{dN_r}{dt} &= f_{cr} \frac{N_c}{\ell_c} - \frac{N_r}{\ell_r} \\
N_c(t) &= N_0 \left[ (1 - R)e^{r_1 t} + (R)e^{r_2 t} \right]
\end{align*}
\]

\[
\begin{align*}
r_j &= \frac{(-1)^j \sqrt{4\ell_c \ell_r (f_{rc}f_{cr} + k_c - 1) + (\ell_c - \ell_r (k_c - 1))^2 - \ell_c + \ell_r (k_c - 1)}}{2\ell_c \ell_r} \\
&= (-1)^j \sqrt{\frac{f' + \alpha}{\ell_r} + \frac{1}{4} \left( \frac{1}{\ell_r} - \alpha \right)^2 + \frac{1}{2} \left( \alpha - \frac{1}{\ell_r} \right)} \\
f' &= \frac{f}{\ell_c} \\
R &= \frac{r_1 - \alpha}{r_1 - r_2}
\end{align*}
\]

\[N_c = \# \text{ of neutrons in fissile core} \quad \ell_c = \text{mean n lifetime in core}\]
\[N_r = \# \text{ of neutrons in reflector} \quad \ell_r = \text{mean n lifetime in reflector}\]
\[k_c = \text{multiplication factor in core} \quad f_{cr} = \text{frac. of core-to-reflector leakage}\]
\[f_{rc} = \text{frac. of reflector-to-core leakage}\]
Goals and Motivations

• **Extract alpha**: While it has been shown that a 2-exp fit is more adequate, we do not yet know how to calculate alpha (or other parameters) from the fit.
  – Rossi-alpha analysis is obsolete otherwise.
  – For most applications, the time between emission and detection (and it’s distribution) is nonnegligible.

• **Estimate Uncertainty**: Currently, we rely on many measurements/long measurements.
  – Having an analytic model will allow uncertainty estimations from a single measurement, ultimately reducing procedural and operational costs.
  – This work also proposes a method of estimating the covariance/correlation between parameters.

• **Application Versatility**: Our first-principles approach will enable adaptation to various applications (including parasitic absorption in the reflector/core).

Probability Density Function Derivation

• We can relate the fission rate \( \frac{dF}{dt} \) to the number of neutrons in the core via the mean time to fission (generation time), \( \tau_f \).

• The number of resulting neutrons is given by multiplying the mean number of neutrons from fission.

• On a case-by-case basis, assuming \( \varepsilon \) is efficiency, the probability of:
  
  i. A fission in \( dt_0 \) about \( t_0 \) is (with average fission rate \( \dot{F}_0 \)):
  
  ii. A count at \( t_1 \) as a result of fission at \( t_0 \) is:

  iii. A count at \( t_2 \) as a result of a count at \( t_1 \) is:

• The probability of a count at \( t_1 \) followed by a count at \( t_2 \) from a common ancestor (not at \( t_j \)) is obtained by integrating the product of (i)-(iii) over \(-\infty < t_0 < t_1 \) and averaging over the distribution of neutrons emitted per fission.

Performing the integration and choosing \( t_i = 0 \) and including a constant term to account for uncorrelated counts yields:

\[
\frac{dF}{dt} = \frac{N_c(t)}{\tau_f}
\]

\[
\frac{dN}{dt} = \bar{\nu} \frac{N_c(t)}{\tau_f} = \bar{\nu} N_0 \frac{dt}{\tau_f} [(1 - R)e^{r_1 t} + Re^{r_2 t}]
\]

i. \( \dot{F}_0 dt_0 \)

ii. \( \varepsilon \nu \frac{dt_1}{\tau_f} [(1 - R)e^{r_1 (t_1 - t_0)} + Re^{r_2 (t_1 - t_0)}] \)

iii. \( \varepsilon (\nu - 1) \frac{dt_2}{\tau_f} [(1 - R)e^{r_1 (t_2 - t_1)} + Re^{r_2 (t_2 - t_1)}] \)

To account for the neutron detected at \( t_1 \)

\[
p(t) = -\frac{\varepsilon \nu (\nu - 1)}{2 \tau_f^2} (e^{r_1 t} \rho_1 + e^{r_2 t} \rho_2) + C
\]

\( \rho_1 \) and \( \rho_2 \) are constant functions of \( R, r_1, \text{and } r_2 \)
Rossi-alpha and Reflector Time

- We now have a PDF!
- This is the function we would use to fit the Rossi-alpha histogram.

- Note: it is **not** the case that one exponent corresponds to the Rossi-alpha while one exponent corresponds to the reflector time.
  - **Reflector Time** is the time between birth and detection (given the neutron is detected).

\[
p(t) = -\frac{\varepsilon v (v - 1)}{2\tau_f^2} (e^{t r_1 \rho_1} + e^{t r_2 \rho_2}) + C
\]

\[
\alpha = (1 - R)r_1 + (R)r_2 \quad \ell_r = -\frac{1}{(R)r_1 + (1 - R)r_2} \quad f' = \frac{f_{cr} f_{rc}}{\ell_c} = -\frac{(1 - R)(R)(r_1 - r_2)^2}{(R)r_1 + (1 - R)r_2}
\]

\[
\rho_1 = \frac{(1 - R)^2}{r_1} + \frac{2(1 - R)(R)}{r_1 + r_2} \quad \rho_2 = \frac{(R)^2}{r_2} + \frac{2(1 - R)(R)}{r_1 + r_2}
\]
Measurement

- HEU (Rocky Flats Shells):
  - 93.12 wt% $^{235}$U
  - Total Mass: 21.8 kg
  - Inner/Outer Radius = 2.013/6.67 cm

- Both systems should measure the same $\alpha$... They do: 1-$\sigma$ intervals overlap

- The $\ell_r$ differ by 26.68 us... We expect < 35 us since outgoing neutrons have already seen poly

### Comparison of measurement systems with differing poly.

<table>
<thead>
<tr>
<th></th>
<th>Bare Tubes</th>
<th>MC-15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1/\alpha$ (us)</td>
<td>$63.9795 \pm 3.7045$</td>
<td>$59.9161 \pm 0.8616$</td>
</tr>
<tr>
<td>$\ell_r$ (us)</td>
<td>$68.40 \pm 2.39$</td>
<td>$95.08 \pm 1.27$</td>
</tr>
</tbody>
</table>
Measurement

- HEU (Rocky Flats Shells):
  - 93.12 wt% $^{235}$U
  - Total Mass: 21.8 kg
  - Inner/Outer Radius = 2.013/6.67 cm

252$^{Cf}$ Interrogation Source

50 cm

Bare $^3$He Tubes

HDPE

NoMAD $^3$He Detection System
(Known Slowing Time = 35-40 us)

Comparison of alpha fit results from 2-exp and 1-exp fits.

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<th>MC-15</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-exp $-1/\alpha$ (us)</td>
<td>63.9795 ± 3.7045</td>
<td>59.9161 ± 0.8616</td>
</tr>
<tr>
<td>1-exp $-1/\alpha$ (us)</td>
<td>171</td>
<td>225</td>
</tr>
</tbody>
</table>

- The alpha’s are (very) different.
- Reflector time is folded into the calculated alphas.
Uncertainty Analysis

1. Our source of uncertainty: what is the time between neutron detections? ⇒ horizontal error bars
2. Horizontal error bars ⇒ which bin does a count belong in?
3. Which bin? ⇒ uncertainty in number of counts per bin ⇒ vertical error bars (which we want!)

We need to describe the above mathematically… Steps:
1. Obtain horizontal error bars;
2. Describe the influence of horizontal error bars on other bins.
3. Estimate the vertical error bars.

*Derivation is more explicit in the conference summary*
Uncertainty Analysis

• Step 1: Obtain Horizontal Error Bars.
  – First, normalize the PDF for each bin \([t_j, t_j + \Delta]\).
  – Second, calculate the mean and standard deviation.

\[
1 = \int_{t_j}^{t_j + \Delta} -\eta(t_j, \Delta) \times A(e^{r_1 t} \rho_1 + e^{r_2 t} \rho_2) dt
\]

\[
p(t, t_j, \Delta) = \eta(t_j, \Delta) p(t) = -\eta(t_j, \Delta) \times A(e^{r_1 t} \rho_1 + e^{r_2 t} \rho_2)
\]

\[
q_i(j) = \int_{t_i}^{t_i+\Delta} \frac{1}{\sqrt{2\pi}\sigma(t_i, \Delta)} \exp \left( -\frac{(\mu(t_i, \Delta) - t)^2}{2\sigma(t_i, \Delta)^2} \right) dt
\]

\[
\sigma(t_j, \Delta) = \int_{t_j}^{t_j+\Delta} t^2 \times p(t, t_j, \Delta) dt - \mu^2
\]

Step 2: Describe the Influence on Other Bins.

– Given \(\mu\) and \(\sigma\) from Step 1, assume a distribution. We assume Normal (Gaussian and Poisson are similar).

– The probability of a count in bin \(j\) belonging in bin \(i\) is equal to the area under the portion of the normal distribution (for bin \(j\) within the boundaries of bin \(i\) (divided by the total area of the normal distribution = 1).
**Uncertainty Analysis**

- **Step 3: Estimate Vertical Error Bars.**
  
  - From Step 2, we have the probability of a bin \( j \) count belonging in bin \( i \), \( q_i(j) \).
  
  - **Sub-step a:** estimate the vertical error bar in bin \( i \) due to bin \( j \) – \( \beta_i(j) \) – by using binomial variance.

- **Step 4: Obtain uncertainty in fit parameters (given \( n \) bins)**
  
  - The covariance matrix of the parameters \( C \) is given by:
    \[
    C = V^{-1} \times \sigma_r^2 \times V
    \]

  - \( V \) is the \( n \times n \) weighting matrix with the \( \beta_i \) on the diagonal.
  
  - \( J \) is the \( n \times ( \text{number of fit parameters} ) \) Jacobian matrix of the fit.
  
  - \( \sigma_r \) is the standard deviation of the residuals.

\[
\beta_i(j) = q_i(j)(1 - q_i(j))p(t, t_j, \Delta)
\]
Preliminary Validation of Uncertainty Analysis

• We took a long measurement of 4.5 kg of a beryllium-reflected plutonium sphere (the BeRP ball).

• The measurement was split into 408 individual files (measurements) and the histograms from each were used to obtain a sample standard deviation.

• The analytic error bars overestimate the uncertainty.

• There is an agreement in the trends. (Excellent at small times/high counts)

• In this analysis, the constant term was subtracted. When working with \(0 \leq \Delta t < \infty\), the constant term must be subtracted for normalization.

• In the future, and all practical cases, the term will not be subtracted when working with individual bins.
Conclusion and Future Work

2-Exp Fit

• The presence of moderating material necessitates a 2-exp treatment to Rossi-alpha analyses.
  – Even systems experiencing non-negligible time-of-flight are better-treated by a 2-exp fit.

• Simulated and measured data will be used to validate the equations.

• The first-principles approach to Rossi-alpha will enable studies of systems exhibiting other phenomena.
  – E.g., fuel containing burnable poisons.

Uncertainty Analysis

• The Jacobian of the fit weighted by the error bars can be used to explicitly calculate the variance and covariance of the fit parameters.

• The error bars can be used to weight the fit of the Rossi-alpha histogram.
  – Error bars depend on the fit; thus, fitting becomes an iterative process until fit parameters converge.

• The analytic error bars predict random error. Coupling analytic to measured uncertainty gives insight to systematic uncertainty.

• Future work also includes further validation.
Acknowledgements

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