2-Exponential PDF and Analytic Uncertainty Approximations for Rossi-alpha Histograms

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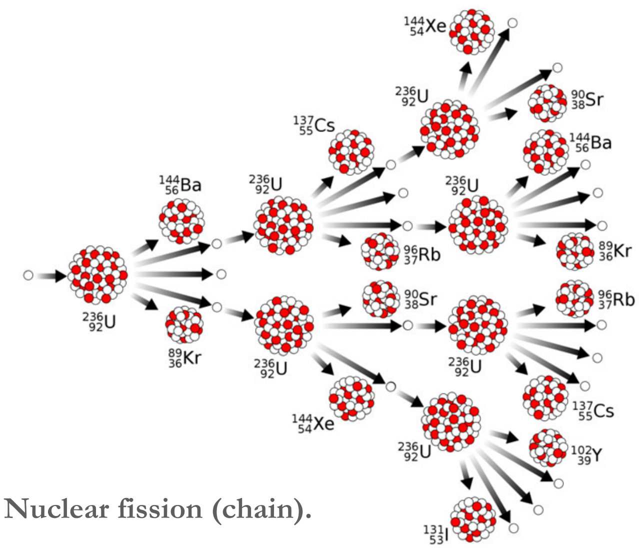
Introduction

- Special nuclear material (SNM) undergoes fission; neutrons from fission can cause another fission…
  - → neutron-multiplying system (characterized by reactivity).

- The reactivity of a subcritical system is of interest in:
  - **Nuclear Nonproliferation**: fuel pin diversion? Is a source
  - neutron-multiplying?
  - **Criticality Safety**: will the reactor regain subcriticality during normal and credible upset conditions? In-situ measurements.
  - **Accelerator-Driven Systems**
  - **Emergency Response**: determine if a sample is multiplying or if neutrons are from another source e.g., (alpha,n).

- **Challenge**: we cannot directly estimate a system's subcritical reactivity.



Nuclear fission (chain).

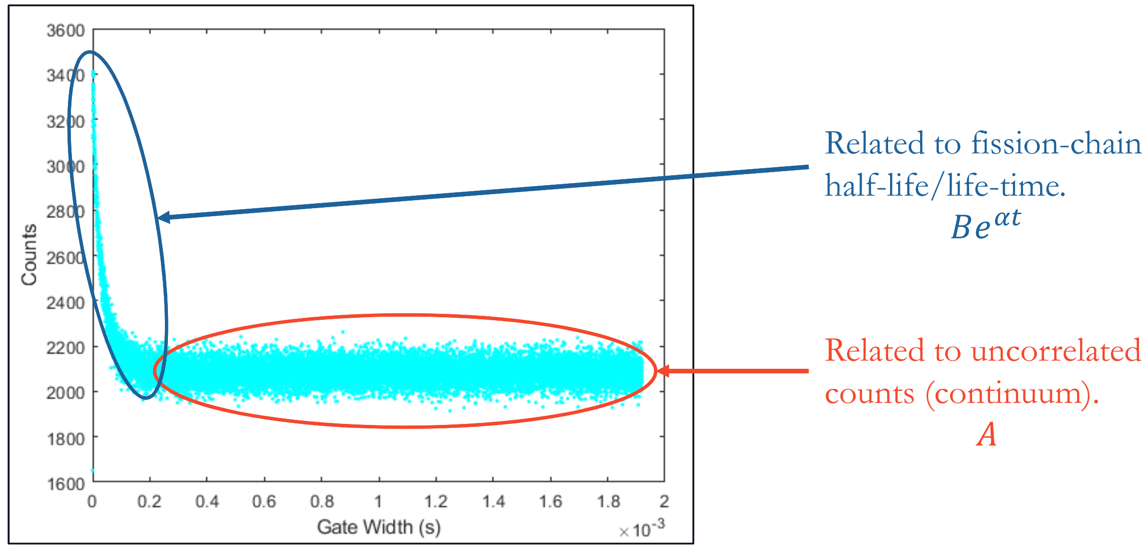
IAEA Inspection

Background

- In neutron-multiplying (fission-chain) systems, neutron emissions/detections are not uniformly distributed in time.
  - This is due to time-correlation between prompt neutrons originating from the same fission.
- The non-uniformity can be observed by producing a histogram of the times between neutron detections.
  - This is the Rossi-alpha histogram.
- Traditionally, it is assumed the trend is described by:

$$p(t) = A + Be^{\alpha t}$$

α is the *prompt-neutron decay constant.*



Sample Rossi-alpha plot.

- The subcritical reactivity can be inferred from the prompt neutron decay constant, alpha.
- We can estimate alpha by fitting the Rossi-alpha histogram with p(t).

Prior Work

- It has been shown for detectors with polyethylene moderation that a 2-exp fit is more adequate than an 1-exp fit due to *slowing-down time.*
  - For this presentation, we assume 2-exp is more adequate and focus on the physical meaning/correspondence of the 2-exp.
- A two-region point-kinetics model for the number of neutrons in a fissile core and reflector has been developed.
- Currently, uncertainty in the estimated Rossi-alpha parameter is calculated by taking <u>many</u> measurements/splitting one <u>long</u> measurement and obtaining a sample standard deviation.

$$\left.\begin{aligned} \frac{dN_c}{dt} &= \frac{k_c - 1}{\ell_c} N_c + f_{rc}\frac{N_r}{\ell_r} \\ \frac{dN_r}{dt} &= f_{cr}\frac{N_c}{\ell_c} - \frac{N_r}{\ell_r} \end{aligned}\right\} \quad N_c(t) = N_0\left[(1-R)e^{r_1 t} + (R)e^{r_2 t}\right]$$

$$r_j = \frac{(-1)^j\sqrt{4\ell_c\ell_r(f_{rc}f_{cr} + k_c - 1) + \left(\ell_c - \ell_{r(k_c-1)}\right)^2} - \ell_c + \ell_r(k_c - 1)}{2\ell_c\ell_r}$$

$$= (-1)^j\sqrt{\frac{f' + \alpha}{\ell_r} + \frac{1}{4}\left(\frac{1}{\ell_r} - \alpha\right)^2} + \frac{1}{2}\left(\alpha - \frac{1}{\ell_r}\right) \qquad f' = \frac{f}{\ell_c}$$

$$\mathrm{R} = \frac{r_1 - \alpha}{r_1 - r_2}$$

N_c = # of neutrons in fissile core

N_r = # of neutrons in reflector

k_c = multiplication factor in core

ℓ_c = mean n lifetime in core

ℓ_r = mean n lifetime in reflector

f_{cr} = frac. of core-to-reflector leakage

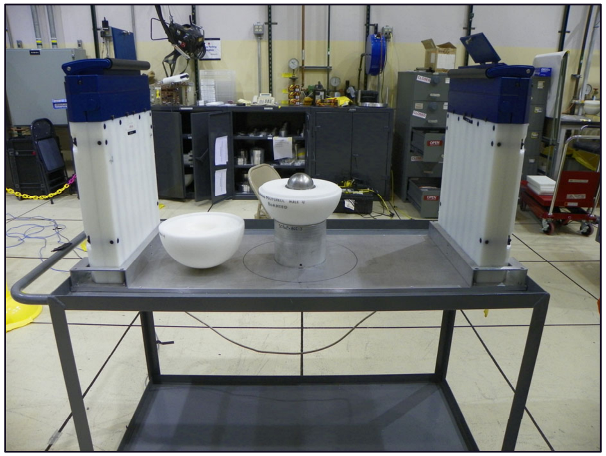
f_{rc} = frac. of reflector-to-core leakage

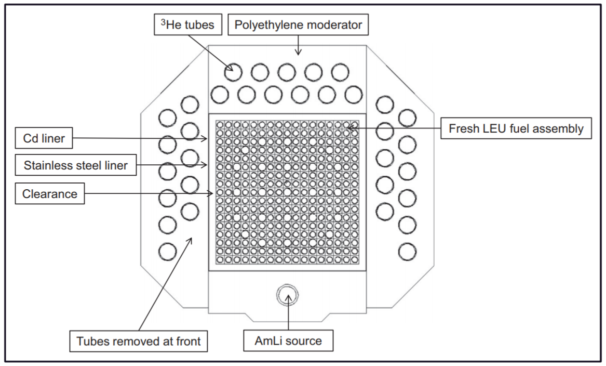
Goals and Motivations

- **Extract alpha**: While it has been shown that a 2-exp fit is more adequate, we do not yet know how to calculate alpha (or other parameters) from the fit.
  - Rossi-alpha analysis is obsolete otherwise.
  - For most applications, the time between emission and detection (and it's distribution) is nonnegligible.

- **Estimate Uncertainty**: Currently, we rely on many measurements/long measurements.
  - Having an analytic model will allow uncertainty estimations from a single measurement, ultimately reducing procedural and operational costs.
  - This work also proposes a method of estimating the covariance/correlation between parameters.

- **Application Versatility**: Our first-principles approach will enable adaptation to various applications (including parasitic absorption in the reflector/core).





L. G. Evans, et al., "A new fast neutron collar for…," *Nuclear Instruments and Methods Section A*, **729**, p740-746, (2013).

Probability Density Function Derivation

- We can relate the fission rate (dF/dt) to the number of neutrons in the core via the mean time to fission (generation time), τ_f.

$$\frac{dF}{dt} = \frac{N_c(t)}{\tau_f}$$

- The number of resulting neutrons is given by multiplying the mean number of neutrons from fission.

$$\frac{dN}{dt} = \bar{\nu}\frac{N_c(t)}{\tau_f} = \bar{\nu}N_0\frac{dt}{\tau_f}[(1-R)e^{r_1 t} + Re^{r_2 t}]$$

- On a case-by-case basis, assuming ε is efficiency, the probability of:
  i. A fission in dt_0 about t_0 is (with average fission rate $\dot{F}_0$):
  ii. A count at t_1 as a result of fission at t_0 is:
  iii. A count at t_2 as a result of a count at t_1 is:

i. $\dot{F}_0 dt_0$

ii. $\varepsilon\nu\frac{dt_1}{\tau_f}\left[(1-R)e^{r_1(t_1-t_0)} + Re^{r_2(t_1-t_0)}\right]$

iii. $\varepsilon(\nu-1)\frac{dt_2}{\tau_f}\left[(1-R)e^{r_1(t_2-t_1)} + Re^{r_2(t_2-t_1)}\right]$

To account for the neutron detected at t_1

- The probability of a count at t_1 followed by a count at t_2 from a common ancestor (not at t_1) is obtained by integrating the product of (i)-(iii) over $-\infty < t_0 < t_1$ and averaging over the distribution of neutrons emitted per fission.

Performing the integration and choosing $t_1 = 0$ and including a constant term to account for uncorrelated counts yields:

$$p(t) = -\frac{\varepsilon\overline{\nu(\nu-1)}}{2\tau_f^2}(e^{r_1 t}\rho_1 + e^{r_2 t}\rho_2) + C$$

ρ_1 and ρ_2 are constant functions of $R, r_1, and\ r_2$

Rossi-alpha and Reflector Time

- We now have a PDF!
- This is the function we would use to fit the Rossi-alpha histogram.

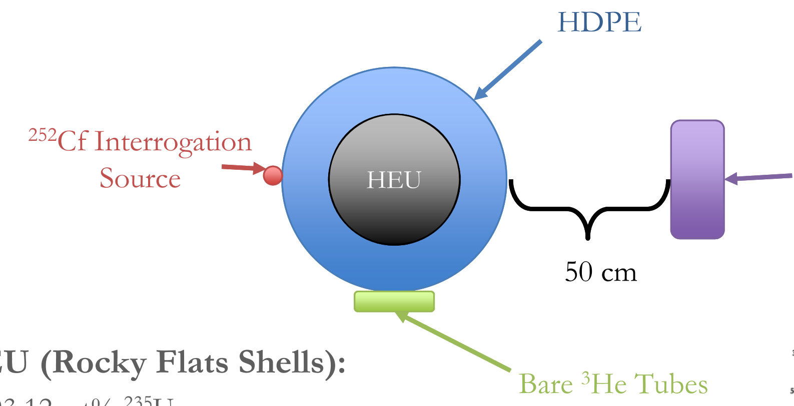
$$p(t) = -\frac{\varepsilon\overline{\nu(\nu-1)}}{2\tau_f^2}\left(e^{r_1 t}\rho_1 + e^{r_2 t}\rho_2\right) + C$$

- Note: it is **not** the case that one exponent corresponds to the Rossi-alpha while one exponent corresponds to the reflector time.
  - Reflector Time is the time between birth and detection (given the neutron is detected).

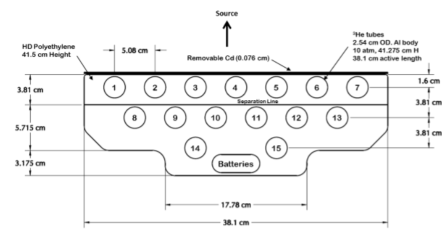
$$\alpha = (1-R)r_1 + (R)r_2 \qquad \ell_r = -\frac{1}{(R)r_1 + (1-R)r_2} \qquad f' = \frac{f_{cr}f_{rc}}{\ell_c} = -\frac{(1-R)(R)(r_1-r_2)^2}{(R)r_1+(1-R)r_2}$$

$$\rho_1 = \frac{(1-R)^2}{r_1} + \frac{2(1-R)(R)}{r_1+r_2} \qquad \rho_2 = \frac{(R)^2}{r_2} + \frac{2(1-R)(R)}{r_1+r_2}$$

Measurement



NoMAD ^{3}He Detection System
(Known Slowing Time = 35-40 us)



- **HEU (Rocky Flats Shells):**
  - 93.12 wt% ^{235}U
  - Total Mass: 21.8 kg
  - Inner/Outer Radius = 2.013/6.67 cm

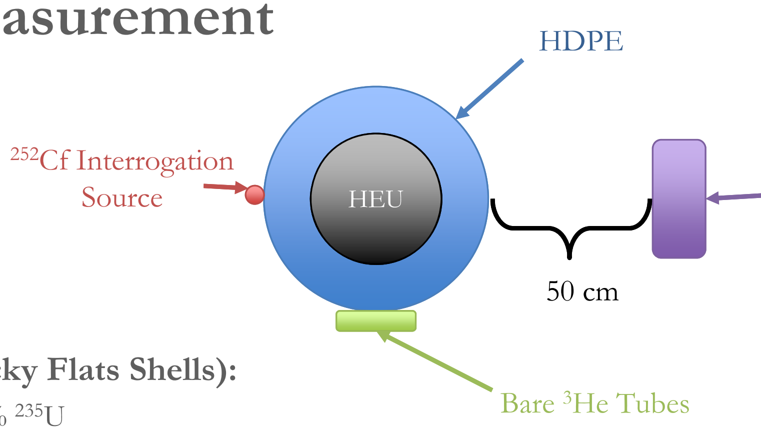
- Both systems should measure the same α… They do: 1-σ intervals overlap

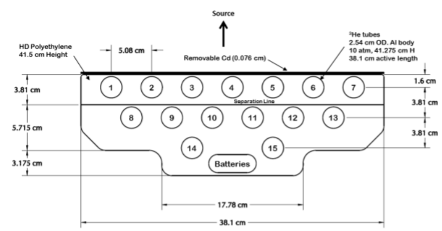
- The ℓ_r differ by 26.68 us… We expect < 35 us since outgoing neutrons have already seen poly

Comparison of measurement systems with differing poly.

	Bare Tubes	MC-15
-1/α (us)	63.9795 ± 3.7045	59.9161 ± 0.8616
ℓ_r (us)	68.40 ± 2.39	95.08 ± 1.27

Measurement





- **HEU (Rocky Flats Shells):**
  - 93.12 wt% ^{235}U
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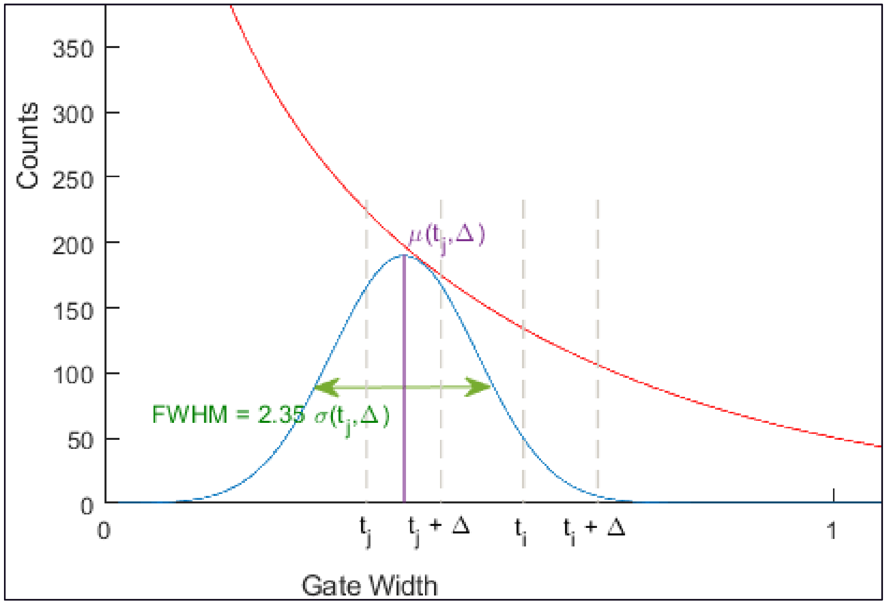
Comparison of alpha fit results from 2-exp and 1-exp fits.

	Bare Tubes	MC-15
2-exp -1/α (us)	63.9795 ± 3.7045	59.9161 ± 0.8616
1-exp -1/α (us)	171	225

- The alpha's are (very) different.
- Reflector time is folded into the calculated alphas.

Uncertainty Analysis

1. Our source of uncertainty: what is the time between neutron detections? ⇒ horizontal error bars
2. Horizontal error bars ⇒ which bin does a count belong in?
3. Which bin? ⇒ uncertainty in number of counts per bin ⇒ vertical error bars (which we want)!



Graphic of horizontal error bars contributing to count uncertainty.

We need to describe the above mathematically… Steps:

1. Obtain horizontal error bars;
2. Describe the influence of horizontal error bars on other bins.
3. Estimate the vertical error bars.

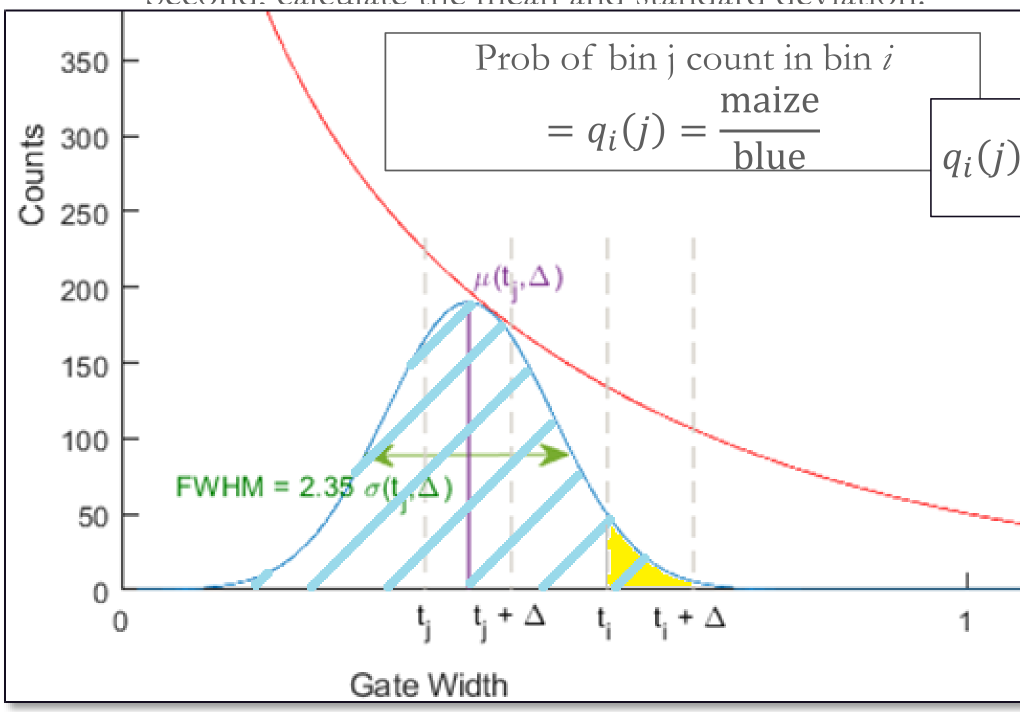
Derivation is more explicit in the conference summary

Uncertainty Analysis

- **Step 1: Obtain Horizontal Error Bars.**
  - First, normalize the PDF for each bin $[t_j, t_j + \Delta]$.
  - Second, calculate the mean and standard deviation.

$$1 = \int_{t_j}^{t_j+\Delta} -\eta(t_j,\Delta) \times A(e^{r_1 t}\rho_1 + e^{r_2 t}\rho_2)dt$$

$$p(t,t_j,\Delta) = \eta(t_j,\Delta)p(t) = -\eta(t_j,\Delta) \times A(e^{r_1 t}\rho_1 + e^{r_2 t}\rho_2)$$



Prob of bin j count in bin i

$$= q_i(j) = \frac{\text{maize}}{\text{blue}}$$

$$q_i(j) = \int_{t_i}^{t_i+\Delta} \frac{1}{\sqrt{2\pi}\sigma(t_j,\Delta)} \exp\left(-\frac{(\mu(t_i,\Delta)-t)^2}{2\sigma(t_i,\Delta)^2}\right) dt$$

$$\sigma(t_j,\Delta) = \int_{t_j} t^2 \times p(t,t_j,\Delta)dt - \mu^2$$

Graphic of horizontal error bars contributing to count uncertainty.

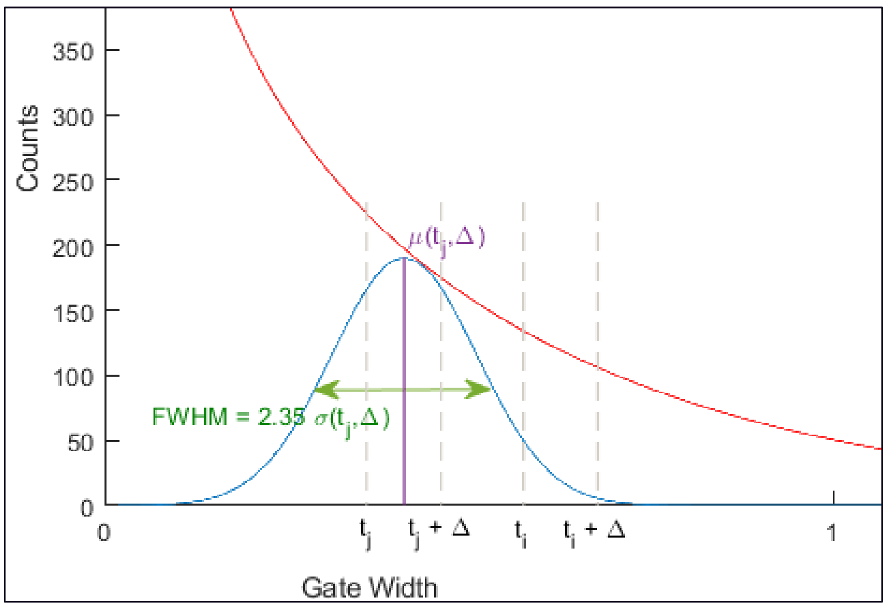
Step 2: Describe the Influence on Other Bins.

- Given μ and σ from Step 1, assume a distribution. We assume Normal (Gaussian and Poisson are similar).
- The probability of a count in bin j belonging in bin i is equal to the area under the portion of the normal distribution (for bin j) within the boundaries of bin i (divided by the total area of the normal distribution = 1).

Uncertainty Analysis

- **Step 3: Estimate Vertical Error Bars.**
  - From Step 2, we have the probability of a bin *j* count belonging in bin *i*, $q_i(j)$.
  - <u>Sub-step a</u>: estimate the vertical error bar in bin *i* due to bin *j* – $\beta_i(j)$ – by using binomial variance.

$$\beta_i(j) = \overbrace{q_i(j)\big(1-q_i(j)\big)}^{\text{Variance for one count}}\overbrace{p(t,t_j,\Delta)}^{\text{Scaling Factor for all counts}}$$



Graphic of horizontal error bars contributing to count uncertainty.

  - <u>Sub-step b</u>: to find the total vertical error bar in bin *i*, β_i, we need to sum the $\beta_i(j)$ over all bins *j*. Then we are done.

$$\beta_i = \sum_{i=0}^{N-1} q_i(j)\big(1-q_i(j)\big)p(t,t_j,\Delta)$$

  - β_i is the 1-σ error bar for bin *i*: $[t_i, t_i + \Delta]$

- **Step 4: Obtain uncertainty in fit parameters (given n bins)**
  - The covariance matrix of the parameters C is given by:

$$V = [J^T W J]^{-1} \qquad C = \sigma_R^2 \times V$$

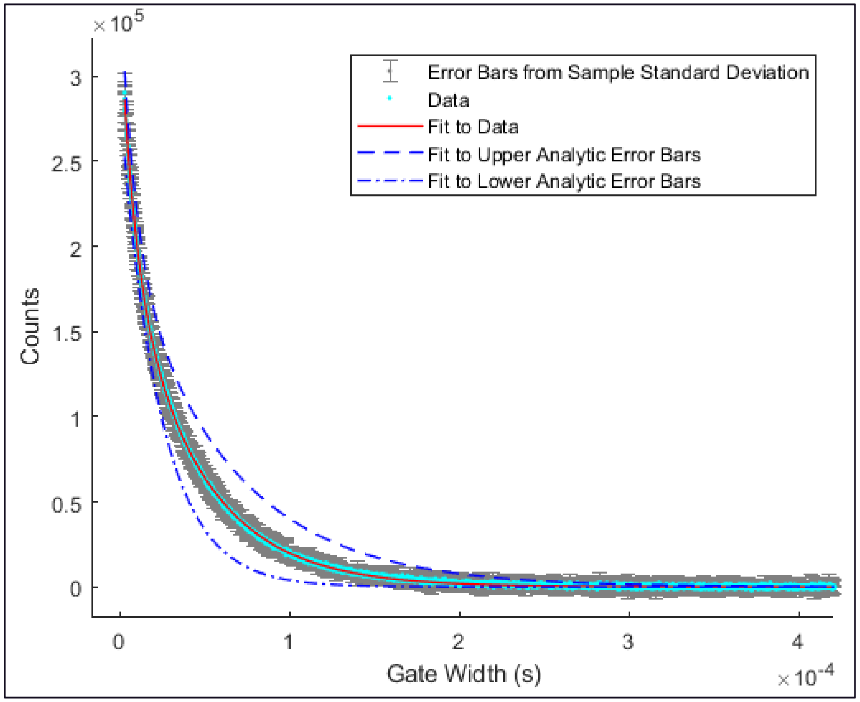
  - W is the n×n weighting matrix with the β_i on the diagonal.
  - J is the n×(number of fit parameters) Jacobian matrix of the fit.
  - σ_R is the standard deviation of the residuals.

Preliminary Validation of Uncertainty Analysis

- We took a long measurement of 4.5 kg of a beryllium-reflected plutonium sphere (the BeRP ball).
- The measurement was split into 408 individual files (measurements) and the histograms from each were used to obtain a sample standard deviation.

- The analytic error bars overestimate the uncertainty.
- There is an agreement in the trends. (Excellent at small times/high counts)

- In this analysis, the constant term was subtracted. When working with $0 \leq \Delta t < \infty$, the constant term must be subtracted for normalization.
- In the future, and all practical cases, the term will not be subtracted when working with individual bins.



Comparison of analytic and sample error bars.

Conclusion and Future Work

2-Exp Fit

- The presence of moderating material necessitates a 2-exp treatment to Rossi-alpha analyses.
  - Even systems experiencing non-negligible time-of-flight are better-treated by a 2-exp fit.
- Simulated and measured data will be used to validate the equations.
- The first-principles approach to Rossi-alpha will enable studies of systems exhibiting other phenomena.
  - E.g., fuel containing burnable poisons.

Uncertainty Analysis

- The Jacobian of the fit weighted by the error bars can be used to explicitly calculate the variance and covariance of the fit parameters.
- The error bars can be used to weight the fit of the Rossi-alpha histogram.
  - Error bars depend on the fit; thus, fitting becomes an iterative process until fit parameters converge.
- The analytic error bars predict random error. Coupling analytic to measured uncertainty gives insight to systematic uncertainty.
- Future work also includes further validation.

Acknowledgements

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